UNIVERSIDAD DE GRANADA

Departamento de
Estadística e
Investigación Operativa

Seminario 2 (SPA Series): Time–inhomogeneous Markov processes and phase–type distributions

17/03/2025

Seminario del profesor **Mogens Bladt** (Department of Mathematical Sciences, University of Copenhagen) **(**

https://researchprofiles.ku.dk/en/persons/mogens-bladt) en la **sala de conferencias del IMAG**. Dos sesiones, el **19 de marzo de 2025**, de **9:30-11:00 y de 11:30 a 13:00**.

1. Introduction to inhomogeneous Markov jump processes and product integration.
2. Inhomogeneous phase-type distributions, IPH.
3. The distribution of rewards.
4. Application to life insurance (survival analysis).
5. Heavy-tailed IPH distributions and insurance risk.
6. Estimation of IPH using the EM algorithm.
7. Stochastic interest rates and IPH.
8. Fitting stochastic interest rates from observed bond prices (IPH fitting)
9. Outlook towards stochastic mortality rates.

We briefly introduce the theory of time–inhomogeneous Markov jump processes using product integration. The principal application of these processes has traditionally been to multi–state life–insurance modelling (introduced in [5]), where, by nature, transition rates are time–inhomogeous (mortality rates e.g. vary according to age). The product integral [6] of a matrix function is the solution to a linear system of differential equations with varying coefficients, and transition probabilities of Markov processes can obtained this way by the well–known Kolmogorov forward and backward equations.

Though time–inhomogeneous phase–type distributions (IPH) [2] are nearly always present as part of multi–state life–insurance models (since the state of death is absorbing), they were never considered as such or treated in their own right. However, they provide a nice addition to standard tools in Markov processes for computing probabilities of first entrance times to different states.

In life insurance, payments are made in the different states of an insured: continuous rates during sojourns or lump sums at transitions. All payments are deterministic and known (by contract). In other fields of Applied Probability, such payments would be known as rewards. The total expected payments (in present value) of a contract is called the reserve. Computing the reserve is hence the same as computing the total expected reward in a time–inhomogeneous Markov process, where the reward structure is as described. One can derive a Laplace transform for this quantity, from which formulas for the expectation (and reserve) and higher order moments can be obtained, [3].

Leaving behind the life–insurance applications, we consider the IPH distributions in more detail. They generalise the well–established time–homogeneous phase–type distributions,[8, 4, 7], and thereby inherit the denseness in the class of distributions on the positive reals, i.e. we may approximate any distribution with positive support arbitrarily well by a IPH distribution. As opposed to ordinary phase–type distributions that all have exponential tails, the tail behaviour of IPH distributions can be almost any. This extension comes with a price: some useful results involving renewal theory and ladder heights are no longer valid for general IPH. However, they can be estimated by an EM algorithm similar to the one applied to ordinary phase-type distributions when only the absorption times are observed (and nothing else). We will show how this algorithm can be applied to estimating heavy-tailed data (insurance risk), light-tailed data (mortality), or fitting stochastic interest rates.

Both in finance and insurance, the calibration of stochastic interest rates plays an important role. The models applied are usually based on solutions to stochastic differential equations (SDE). Instead, we consider a dense class of stochastic interest rate models (so-called Markovian interest rate models) in which an underlying Markov jump process dictates the type of a (deterministic or constant) spot rate at different times. This model goes back at least to [9]. It turns out that this model is intimately connected to phase–type distributions: the bond price in a Markovian interest model equals the survival function of a phase–type distribution. This connection, which was not noticed until recently, [1], allows us to provide a dense class of stochastic interest rate models that can be calibrated to data (observed bond prices), and which also fits perfectly into the life insurance set-up.

In the seminar, we will demonstrate how to perform the fitting using R (with the package ”ma-trixdist”). The R programs will be made available.

References

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